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Note

On the Condition $\sum_{n=1}^{\infty} n^{p-1}E_n^*(f) < \infty$

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For every integer $k \geqslant 0$ let C^{*k} denote the set of 2π-periodic real functions having a continuous kth derivative on $(-\infty, \infty)$. C^* will mean C^{*0} and $\| \ \|$ will denote the sup norm over $[-\pi, \pi]$. For $n = 1, 2,...$ and every $f \in C^*$ let

$$E_n^*(f) = \min \|f - t_n\|,$$

where the minimum is taken over all trigonometric polynomials t_n of order $\leqslant n$. Let p be a fixed integer $\geqslant 1$. Suppose that for some $f \in C^*$,

$$\sum_{n=1}^{\infty} n^{p-1}E_n^*(f) < \infty.$$

Then [1, Theorem 8, p. 61] $f \in C^{*p}$. Thus if $c_1, c_2,...$ are positive numbers with

$$\sum_{n=1}^{\infty} c_n < \infty, \tag{1}$$

then $f \in C^{*p}$ whenever

$$f \in C^* \qquad \text{and} \qquad n^{p-1}E_n^*(f) = O(c_n). \tag{2}$$

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According to [2, Sect. 18, Theorem], (1) is essential for the validity of the last statement. Namely, if

$$c_1, c_2, \ldots > 0, \qquad \sum_{n=1}^{\infty} c_n = \infty \tag{3}$$

and

$$(c_n/n^{p-1})_{n=1}^{\infty} \text{ is nonincreasing,} \tag{4}$$

then there is an $f \notin C^{*p}$ satisfying (2).

Suppose (3) and

$$(nc_n)_{n=1}^{\infty} \text{ is nonincreasing;} \tag{5}$$

then (4) and hence there is an f as in the last sentence. The purpose of this note is to make a stronger statement, i.e.,

THEOREM. *Assume* (3) *and* (5). *Then there exists an* f *satisfying* (2) *which does not have throughout* $(-\infty, \infty)$ *a* $(p-1)$th *derivative satisfying in* $[-\pi, \pi]$ *a Lipschitz condition.*

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We shall need the following

LEMMA. *Let* f *be a* 2π*-periodic real function for which* $f^{(p-1)}$ *exists on* $(-\infty, \infty)$ *and satisfies in* $[-\pi, \pi]$ *a Lipschitz condition. For* $n = 1, 2, \ldots$ *let* τ_n *be a real trigonometric polynomial of order* $\leqslant n$ *for which* $\|f - \tau_n\| = O(n^{-p})$. *Then* $\|\tau_n^{(p)}\|_{n=1}^{\infty}$ *is bounded.*

Proof of the Lemma. Ler r be the smallest integer $\geqslant (p+2)/2$ and let n be a positive integer. Set

$$\begin{aligned} s_n(x) &= (\sin\{([n/r]+1)x/2\}\{\sin(x/2)\}^{-1})^{2r}, && \text{if} \quad -\pi \leqslant x \leqslant \pi,\ x \neq 0, \\ &= ([n/r]+1)^{2r}, && \text{if} \quad x = 0, \end{aligned}$$

so that s_n is continuous at 0, and let

$$K_n(x) = s_n(x) \Big/ \int_{-\pi}^{\pi} s_n, \qquad -\pi \leqslant x \leqslant \pi,$$

$$I_n(x) = -\int_{-\pi}^{\pi} K_n(t) \sum_{k=1}^{p} (-1)^k \binom{p}{k} f(x+kt)\, dt, \qquad -\infty < x < \infty \tag{6}$$

([ ] denotes integral part). Then

$$\|f - I_n\| \leqslant M n^{1-p} \omega(1/n) \leqslant M_1 n^{-p}, \tag{7}$$

where ω is the modulus of continuity of $f^{(p-1)}$, M and M_1 are independent of n [1, pp. 56–58]; and

$$I_n^{(p-1)}(x) \equiv - \int_{-\pi}^{\pi} K_n(t) \sum_{k=1}^{p} (-1)^k \binom{p}{k} f^{(p-1)}(x + kt)\, dt.$$

Furthermore, since $f^{(p-1)}$ satisfies a Lipschitz condition on $(-\infty, \infty)$ and hence is almost everywhere differentiable there, we have there [3, Sect. 39.1s, p. 216]

$$I_n^{(p)}(x) = - \int_{-\pi}^{\pi} K_n(t) \sum_{k=1}^{p} (-1)^k \binom{p}{k} f^{(p)}(x + kt)\, dt,$$

the integral being Lebesgue. Since

$$\int_{-\pi}^{\pi} |K_n| = \int_{-\pi}^{\pi} K_n = 1,$$

we have

$$\|I_n^{(p)}\| \leqslant M_2, \tag{8}$$

M_2 being independent of n.

Also

$$K_n(t) = \sum_{k=0}^{n} a_k^{(n)} \cos kt, \qquad -\pi \leqslant t \leqslant \pi,$$

where $a_k^{(n)}$ are real constants, and therefore by (6), $I_n(x)$ is a real trigonometric polynomial of order $\leqslant n$ [1, pp. 57–58].

By Bernshtein's inequality for trigonometric polynomials and (8), for $n = 1, 2, \ldots$,

$$\begin{aligned} \|\tau_n^{(p)}\| &\leqslant \|\tau_n^{(p)} - I_n^{(p)}\| + \|I_n^{(p)}\| \leqslant n^p \|\tau_n - I_n\| + M_2 \\ &\leqslant n^p(\|f - \tau_n\| + \|f - I_n\|) + M_2 \end{aligned}$$

which is bounded by hypothesis and (7).

3. Proof of the Theorem

Let

$$f(x) \equiv \sum_{k=0}^{\infty} 5^{(1-p)k} c_{5^k} \cos 5^k x.$$

Observe that the series converges uniformly on $(-\infty, \infty)$ as, for $k = 0, 1, 2, \ldots,$

$$5^{(1-p)k} c_{5^k} = 5^k c_{5^k} 5^{-pk} \leqslant c_1 5^{-pk}$$

by (5).

Let n be an integer $\geqslant 1$. Define j by

$$5^j \leqslant n < 5^{j+1}, \qquad j \text{ an integer}.$$

Then

$$\left.\begin{aligned} E_n^*(f) &\leqslant \left\| f(x) - \sum_{k=0}^{j} 5^{(1-p)k} c_{5^k} \cos 5^k x \right\| = \sum_{k=j+1}^{\infty} 5^k c_{5^k} 5^{-pk} \\ &\leqslant 5^{j+1} c_{5^{j+1}} \sum_{k=0}^{\infty} 5^{-p(j+1)} 5^{-pk} < 5^{j+1} c_{5^{j+1}} n^{-p} \sum_{k=0}^{\infty} 5^{-pk} \\ &\leqslant \tfrac{5}{4} n c_n n^{-p} \leqslant \tfrac{5}{4} c_1 n^{-p}. \end{aligned}\right\} \tag{9}$$

Hence (2).

Set $\tau_n(x) \equiv \sum_{k=0}^{j} 5^{(1-p)k} c_{5^k} \cos 5^k x$ so that, as $n \to \infty$,

$$\| f - \tau_n \| = O(n^{-p}).$$

If p is odd, then, for every real x, $|\tau_n^{(p)}(x)| = |\sum_{k=0}^{j} 5^k c_{5^k} \sin 5^k x|$ and since, for $k = 0, 1, 2, \ldots, 5^k$ is congruent to 1 (mod 4),

$$\| \tau_n^{(p)} \| = \sum_{k=0}^{j} 5^k c_{5^k} \sin 5^k (\pi/2) = \sum_{k=0}^{j} 5^k c_{5^k}.$$

If p is even, then again $\| \tau_n^{(p)} \| = \sum_{k=0}^{j} 5^k c_{5^k}$. By (5), $(c_n)_{n=1}^{\infty}$ is decreasing. By Cauchy's condensation test and (3), $\sum_{k=0}^{\infty} 5^k c_{5^k} = \infty$. Hence, for $j = 0, 1, 2, \ldots,$

$$\| \tau_{5^j}^{(p)} \| = \sum_{k=0}^{j} 5^k c_{5^k} \to \infty$$

so that $\| \tau_n^{(p)} \|_{n=1}^{\infty}$ is unbounded.

If f had throughout $(-\infty, \infty)$ a $(p-1)$th derivative satisfying in $[-\pi, \pi]$ a Lipschitz condition, then, by the lemma, $\|\tau_n^{(p)}\|_{n=1}^{\infty}$ would be bounded.

Remark 1. Note that this $f \in C^{*p-1}$ as, by the last inequality in (9), $\sum_{n=1}^{\infty} n^{p-2} E_n^*(f) < \infty$.

Remark 2. The first $\leqslant$ sign in (9) can be replaced by $=$ [4, Sect. 2.11.2, p. 77].

References

1. G. G. Lorentz, "Approximation of Functions," Holt, New York, 1966.
2. S. N. Bernshtein, On the best approximation of continuous functions by polynomials of a given degree, *in* "Collected Works," Vol. I, translated by the U.S. Atomic Energy Commission, AEC-tr-3460, 1958. The original paper has appeared in the Communications of the Kharkov Mathematical Society in 1912.
3. E. J. McShane, "Integration," Princeton Univ. Press, Princeton, N.J., 1944.
4. A. F. Timan, "Theory of Approximation of Functions of a Real Variable," MacMillan, New York, 1963.